King Fahd University of Petroleum & Minerals
Information & Computer Science Department

ICS 253: Discrete Structures I

Advanced Counting Techniques

Reading Assignment

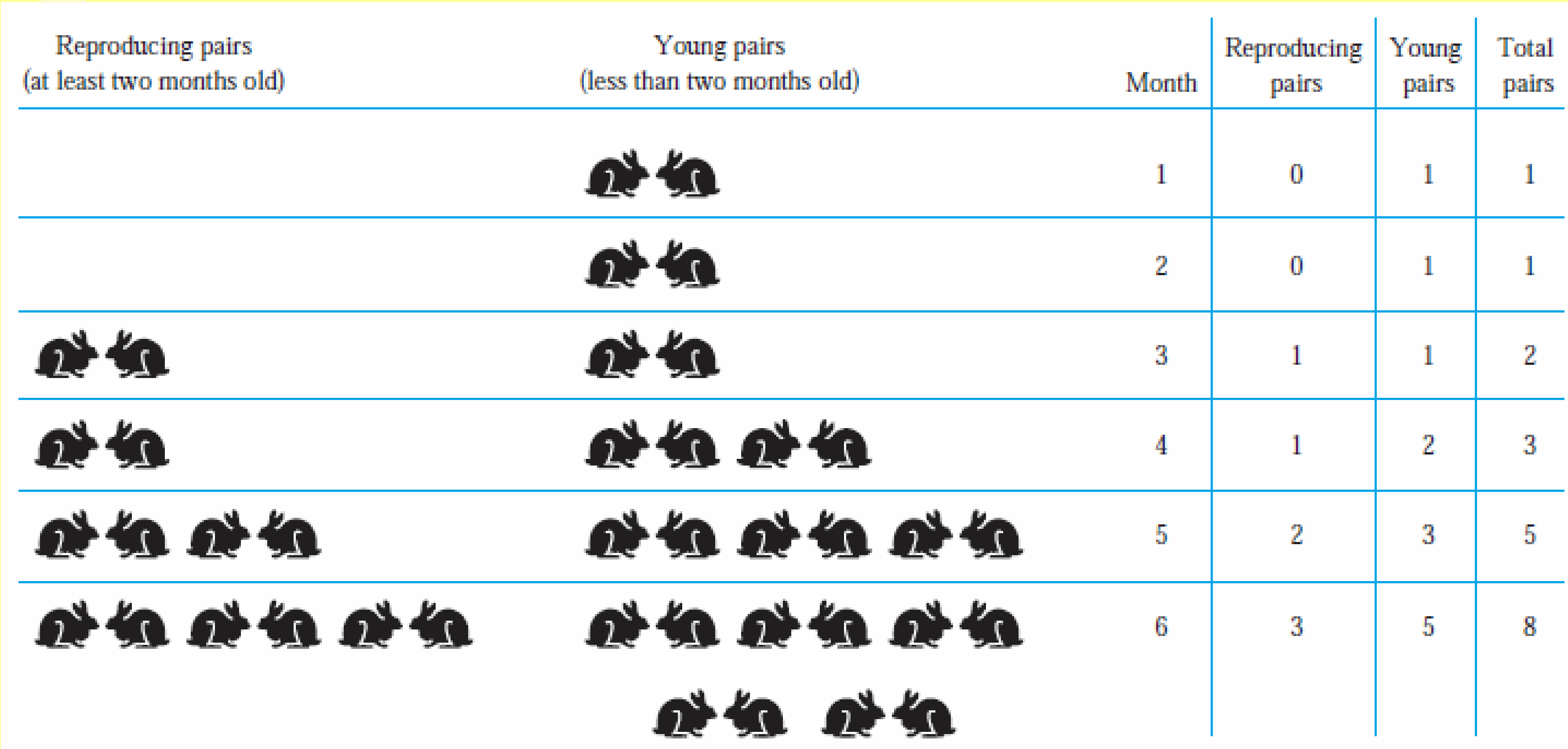
- K. H. Rosen, *Discrete Mathematics and Its Applications*, 6th Ed., McGraw-Hill, 2006.
  - Chapter 8 (Sections 8.1 and 8.2 up to the end of page 502)

Section 8.1: Applications of Recurrence Relations

- Many counting problems cannot be solved easily using the methods discussed in Chapter 6.
- We will see how recurrences can be used to solve such problems.

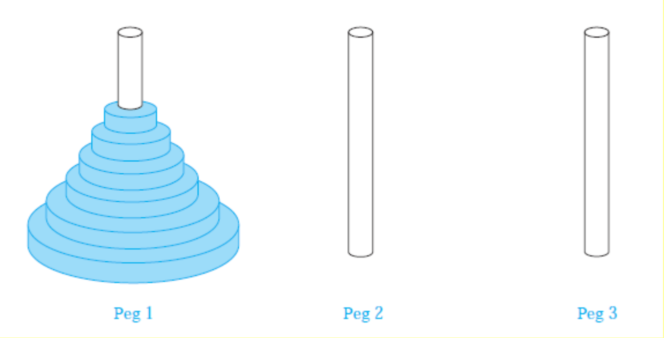
Example 1

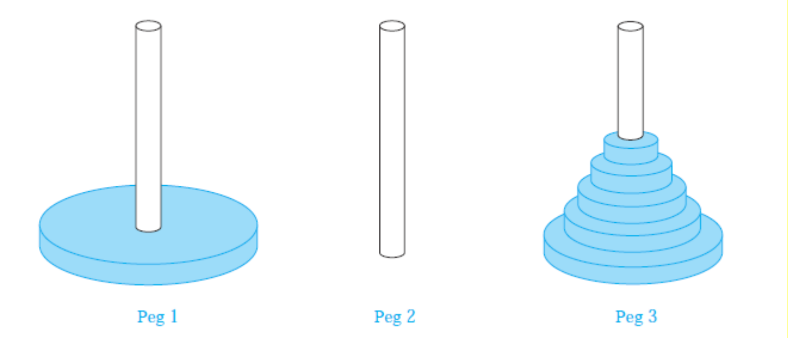
- A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month, as shown in the Figure. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.



Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Towers of Hanoi





Examples

- Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length five?

Examples

- Q5 Page 494: Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.
  - What are the initial conditions?
  - How many bit strings of length seven contain two consecutive 0s?

Examples

- A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n.

Examples

- Find a recurrence relation for C_n, the number of ways to parenthesize the product of n + 1 numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n$, to specify the order of multiplication.
  - What is C2? C3?
  - This sequence is called the Catalan Numbers.

Section 8.2:Solving Linear Recurrence Equations

- We will only consider solving linear homogeneous recurrence equations with constant coefficients.
  - Linear nonhomogeneous recurrence relations with constant coefficients are ommitted.
  - It will be enough for us, especially those related to estimating the complexity of recursive algorithms.

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

- A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients (k=2)

Theorem 1:

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Theorem 2:

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Example 1

- Q2, pp 508: Solve these recurrence relations together with the initial conditions given.

  (d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$.

Example 2

- Q2, pp 508: Solve these recurrence relations together with the initial conditions given.

  (f) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$.

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

- Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$$

for n=0,1, 2, … where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.